Tensor Completions of CSA-Groups and Fraisse Limits of Extensions of Centralizers

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Outline

- Groups with exponents in a ring
- Magnus' free $\mathbb{Q}$-group
- Lyndon's axioms and free $\mathbb{Z}[t]$-group
- Completions
- Friasse limits
- Non-standard models of free groups

Examples of groups with exponentiation

Let G be a group, R be a unitary ring.

An R-group (group with R-exponentiation) G is a group G equipped with an R-action $(g, r) \to g^r$ that satisfy some conditions.

- every group G admits $\mathbb{Z}$-exponentiation: $(g, n) \to g^n$,
- R-modules over a ring R,
- divisible groups are $\mathbb{Q}$-groups,
- pro-p-groups are $\mathbb{Z}_p$-groups, where $\mathbb{Z}_P$ is the ring of p-adic integers,
- profinite groups are $\hat{\mathbb{Z}}$-groups, where $\hat{\mathbb{Z}}$ is the completion of $\mathbb{Z}$ in the profinite topology.

R-groups as "non-commutative" R-modules

R-groups play a role similar to R-modules in the class of abelian groups.

In fact, one can think of R-groups as of "non-commutative" R-modules.

Divisible groups

A group G is called divisible (or a $\mathbb{Q}$-group) if every equation of the type

$$x^n = g$$

has a unique solution in G for every $g \in G$ and $n \in \mathbb{Z}$.

Divisible abelian groups $=$ $\mathbb{Q}$-vector spaces.

By now the theory of divisible groups is more than 100 years old.

For every group G there is a canonical homomorphism

$$G \to G^{\mathbb{Q}}$$

into the divisible hull $G^{\mathbb{Q}}$ of G (or $\mathbb{Q}$-completion of G).

$G^{\mathbb{Q}}$ is a $\mathbb{Q}$-group which is uniquely defined by G (up to $\mathbb{Q}$-isomorphisms).

More precisely, $G^{\mathbb{Q}}$ is a unique $\mathbb{Q}$-group (up to $\mathbb{Q}$-isomorphism) which satisfies the following universal property:

Free $\mathbb{Q}$-groups

Let $F = F(X)$ be a free group with basis X.

Then $F^{\mathbb{Q}}$ is a free $\mathbb{Q}$ group with basis X, i.e., the free object in the category of divisible groups.

Theorem [Baumslag, 1968]

The group F^{Q} is a direct limit of a countable chain of free root extensions.

Recall that a free root extension of an element $g \in G$ is a free product with amalgamation:

$$G *_{g=x^n} \langle x \rangle = \langle G, x \mid g = x^n \rangle.$$

The group $F^{\mathbb{Q}}$ enjoys nice algorithmic properties.

Theorem [Kharlampovich-M.]

The Diophantine problem is decidable in $F^{\mathbb{Q}}$.

Recall that the decidability of the Diophantine problem means that there is an algorithm that for any finite system of group equations with coefficients in $F^{\mathbb{Q}}$ decides whether or not it has a solution in $F^{\mathbb{Q}}$, and if so, it finds a solution.

Magnus homomophism

Let $X = \{x_1, \ldots, x_n\}$ and consider the map

$$x_i \longrightarrow (1 + x_i)$$

from the generators of F^Q into the formal power series ring $Q\langle\langle x_1, ..., x_n\rangle\rangle$ with coefficients in Q.

It is known that this map induces a homomorphism

$$\lambda : F^Q \longrightarrow Q\langle\langle x_1, ..., x_n\rangle\rangle$$

the Magnus homomorphism.

Long-standing Magnus problem

Is λ injective? Or, equivalently, is the group F^Q residually torsion-free nilpotent?

Magnus homomophism

Theorem [Jaikin-Zapirain, 2021]

The Magnus homomorphism is injective.

One can apply Lie methods to the group $F^{\mathbb{Q}}$.

Let G be a group.

$\hat{G}$ is the profinite completion of G, i.e., the inverse limit of all finite quotients of G.

$\hat{G}_p$ is the pro-p-completion of G, i.e., the inverse limit of all finite p-quotients of G.

The groups $\hat{F}$ and $\hat{F}_p$ are, correspondingly, free profinite and free pro-p-groups.

However, I doubt that these are free $\mathbb{Z}_p$-groups.

Qpen Problem

Show that $\hat{F}$ and $\hat{F}_p$ are not, correspondingly, free profinite or free pro-p-groups.

The following problem attracted a lot of attention.

Qpen Problem [Remeslennikov]

Let F be a free non-abelian group of finite rank and G a finitely generated residually finite group. Is it true that

$$\hat{F} \simeq \hat{G} \iff F \simeq G?$$

Axioms of R-groups

Let R be an associative unitary ring and G a group. Consider the following axioms on an exponentiation $(g, \alpha) \to g^\alpha$:

1) $g^1 = g, \quad g^0 = 1, \quad 1^\alpha = 1,$
2) $g^{\alpha+\beta} = g^\alpha g^\beta, \quad g^{\alpha\beta} = (g^\alpha)^\beta,$
3) $(h^{-1}gh)^\alpha = h^{-1}g^\alpha h,$
4) $gh = hg \longrightarrow (gh)^\alpha = g^\alpha h^\alpha.$

Definition

G is an R-group if it satisfies the axioms 1)-4).

In fact, in 1960 Lyndon introduced only axioms 1)-3), even though all the R-groups he considered satisfy 4).

Later Miasnikov and Remeslennikov added the axiom 4), so all abelian R-groups are R-modules.

Axioms of R-groups

For each fixed ring R the class of all R-groups is a quasi-variety, so there exist free objects, called free R-groups.

All the groups we discussed in this talk satisfy the axioms 1)-4), so they are R-groups for the corresponding rings R.

Furthermore, the group $F^{\mathbb{Q}}$ is, indeed, a free $\mathbb{Q}$-group, but $\hat{F}$ is probably not.

Definition

Let G be a group and R an associative unitary ring. An R-group G^R is called a tensor R-completion or simply R-completion of G if there is a homomorphism $\lambda : G \to G^R$ such that:

- $\lambda(G)$ R-generates G^R,
- for any R-group H and any homomomorphism $\phi : G \to H$ there exists an R-homomorphism $\psi : G^R \to H$ such that $\phi = \psi \circ \lambda$, i.e., the following diagram is commutative

$$\begin{tikzcd} G \arrow[rr, "\lambda"] \arrow[dr, "\phi"'] & & G^R \arrow[dl, dashed, "\exists\,\psi"] \\ & H & \end{tikzcd}$$

The following results are standard.

- For every group G and every ring R there exists an R-completion G^R.
- The completion G^R is unique up to an R-isomorphism.

The homomorphism $\lambda_R : G \to G^R$ is called the canonical one.

R-completion $\lambda_R : G \to G^R$ is called faithful if λ_R is injective.

The R-completions F^R are, indeed, free R-groups and the canonical homomorphism $F \to F^R$ is injective.

Lyndon'sgroup $F^{\mathbb{Z}[t]}$

In 1960 Lyndon introduced a free $\mathbb{Z}[t]$-group $F^{\mathbb{Z}[t]}$ to study equations and algebraic geometry in free groups F.

He treated elements of $F^{\mathbb{Z}[t]}$ as parametric words, where together with the standard multiplication one is allowed to take exponentiation by polynomials from $\mathbb{Z}[t]$, such as

$$(x^{f(t)}y^{g(t)})^{h(t)}u^{s(t)},$$

modulo the congruence generated by the consequences of the axioms.

Algebraic sets in F

His idea was to describe the solution sets A of finite system of equations $S(x_1, \ldots, x_n, F) = 1$ in F as finite unions of parametric sets

$$A = P_1 \cup \ldots \cup P_n,$$

where P_i is the set of all values of some parametric word $w_i \in F^{\mathbb{Z}[t]}$ under all specialization homomorphisms $\phi_n : F^{\mathbb{Z}[t]} \to F$ induced by the homomorphism $\mathbb{Z}[t] \to \mathbb{Z}$ that maps $t \to n$, $n \in \mathbb{Z}$.

In particular, the parametric set defined by a parametric word u^t, where $u \in F$ is the cyclic subgroup generated by u.

It follows that equation $xu = ux$ in F, where $u \in F$ is not a proper power is described precisely by trhe parametric word u^t.

Lyndon showed that the solution sets of one-variable equations are indeed finite unions of parametric sets.

In general his idea does not hold, but his intuition was not far off as the following result shows.

Theorem [Kharlampovich-M.]

Every algebraic set (the solution set of a finite system of equations) over F can be obtained as the Zariski closure of a finite union of parametric sets.

We obtained a much more precise description of algebraic sets, but there is no time to explain it here.

Discriminating homomorphisms

Lyndon's proofs were technically quite challenging because he did not have a clear algebraic description of the structure of the group $F^{\mathbb{Z}[t]}$.

Nevertheless, he proved that the set of specialization homomorphisms

$$\Phi = \{\phi_n \mid n \in \mathbb{Z}\}$$

discriminates $F^{\mathbb{Z}[t]}$ into F, i.e., for any finite set of elements $E \subset F^{\mathbb{Z}[t]}$ there is $\phi_n \in \Phi$ such that the restriction of ϕ_n on E is injective.

Like in the classical algebraic geometry to understand solution sets of finite systems of equations in a group one has to study the coordinate groups of such systems.

Zariski topology (generated by the algebraic sets as prebasis of closed sets) over free groups is Noetherian, hence every algebraic set is a finite union of irreducible ones.

This implies that it suffices to study only the coordinate groups of the irreducible algebraic sets.

The following result is rather general, but I state it just for free groups.

Theorem [Baumslag-M.-Remeslennikov]

Let G be a finitely generated group. Then the following conditins are equivalent:

- G is the coordinate group of an irreducible algebraic set over F;
- G is universally equivalent to F;
- G is discriminated by F.

Combining this with Lyndon's result on discrimination of $F^{\mathbb{Z}[t]}$ one immediately gets a large source of the coordinate groups of irreducible algebraic sets over F - all finitely generated subgroups of $F^{\mathbb{Z}[t]}$.

Unfortunately, Lyndon's description of the group $F^{\mathbb{Z}[t]}$ as equivalence classes of "parametric words" does not give much understanding on subgroups of the group.

Theorem [M.-Remeslennikov, 1996]

The group $F^{\mathbb{Z}[t]}$ is union of an infinite countable chain $F = G_0 < G_1, \ldots$ of groups such that G_{i+1} is an extension of a centralizer in G_i, i.e.,

$$G_{i+1} = \langle G_i, t_i \mid [t_i, C_i] = 1 \rangle$$

for some centralizer C_i from G_i.

Note, that these extensions of centralizers are very particular cases of free constructions (HNN extensions and free products with amalgamation).

Subgroups of $F^{\mathbb{Z}[t]}$

By that time Bass-Serre theory was already developed. By design, it allows one to describe subgroups of free constructions as fundamental groups of graphs of groups.

In particular, it gives description of finitely generated subgroups of $F^{\mathbb{Z}[t]}$ as the fundamental groups of particularly nice graphs of groups.

How about other groups discriminated by F, which are not subgroups of $F^{\mathbb{Z}[t]}$?

Subgroups of $F^{\mathbb{Z}[t]}$

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How about other groups discriminated by F, which are not subgroups of $F^{\mathbb{Z}[t]}$?

Theorem [Kharlampovich and M.]

Every finitely generated group discriminated by F embeds into $F^{\mathbb{Z}[t]}$.

This is technically a very demanding result that requires some essential development of Makanin-Razborov technique on solving equations in free groups, in particular, a novel description of solution sets of systems of equations in free groups via triangular quasi-quadratic systems.

Generalizations

In fact, all the results about free groups that I discussed here (except for Zaikin-Zapirain theorem) hold in a much more general class of groups.

In particular, in arbitrary torsion-free hyperbolic groups.

These were crucial for solving Tarski problems for free and torsion-free hyperbolic groups.

Generalizations

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In particular, in arbitrary torsion-free hyperbolic groups.

These were crucial for solving Tarski problems for free and torsion-free hyperbolic groups.

Now I will describe algebraic structure of G^R for a very wide class of groups G. For this I need a few defintions.

Definition

A subgroup H of a group G is called conjugately separated or malnormal if $H \cap H^x = 1$ for any $x \in G \smallsetminus H$.

Definition

A group G is called a CSA-group if all its maximal abelian subgroups are conjugately separated.

Free, torsion-free hyperbolic, and many other groups are CSA. This is a truly large class.

It follows from the properties of tensor products of modules, that if M is torsion-free abelian group and R has characteristic zero then:

- R-module $M \otimes R$ is the R-completion of M,
- the canonical map $M \to M \otimes R$ is injective.

From now on for simplicity I will consider only torsion-free groups G and rings R of characteristic zero.

Tensor extensions of abelian subgroups

Let G be a group and M a maximal abelian subgroup of G. Then amalgamated free product

$$G(M,R) = \langle G * (M \otimes R) \mid M = i(M) \rangle,$$

where $i : M \to M \otimes R$ is the canonical embedding, is called the tensor extension of M by R.

Theorem [M.-Remeslennikov, 1996]

Let G be a torsion-free CSA group and R a ring of characteristic zero. Then:

- G^R is a union of a chain of tensor extensions of centralizers;
- the canonical homomorphism $G \to G^R$ is inective;
- the group G^R is torsion free and CSA.

Theorem [Kharlampovich, M., Sklinos, 2020]

Let G be a torsion-free hyperbolic group then the Lyndon's completion $G^{\mathbb{Z}[t]}$ is a Friasse limit of iterated extension of centralizers of G. In particular, $F^{\mathbb{Z}[t]}$ is a Friasse limit of iterated extension of centralizers of F.

This result implies various universal and homogeneous properties of $F^{\mathbb{Z}[t]}$.

Fraisse limits of tensor extensions of centralizers

Theorem [Amaglobeli-M.]

Let G be a torsion-free CSA group and R an associative unitary ring of characteristic zero. Then:

- the class $\mathcal{C}$ of iterated tensor R-extension of centralizers of G forms a Fraisse category;
- the R-completion G^R of G is the Fraisse limit of $\mathcal{C}$.

Tarski problems in free groups

Tarski problems for free groups F were solved by Kharlampovich and Miasnikov, and, independently, by Sela:

- $Th(F)$ is decidable
- All nonabelian free groups have the same first-order theory ($Th(F_n) = Th(F_m)$ for all $m, n \geq 2$)
- $Th(F)$ has effective quantifier elimination to $\forall\exists$-formulas.

First-order classification problem

First-order classification problem

For a given group G classify all groups H which are first-order equivalent to G, i.e., they satisfy precisely the same first-order sentences as G (symbolically $G \equiv H$).

I will discuss this problem only for a non-abelian free group F.

First-order classification for free groups

A group G is called elementary free if $G \equiv F$, where F is a free non-abelian group.

Finitely generated case is done:

Theorem [Kharlampovich-M., Sela, 2006]

Let F be a nonabelain free group. Regular NTQ groups (ω-residually free towers, hyperbolic towers) are exactly the finitely generated elementary free groups.

Theorem [Kharlampovich-Natoli]

Let G be a countable elementary free group in which all abelian subgroups are cyclic. Then G is a union of a chain of finitely generated elementary free groups.

This is based on Kharlampovich-M.-Sklinos paper on Fraisse limits of limit groups.

However, the example $\mathbb{Z} * (\mathbb{Z} + \mathbb{Q})$ shows that not every countable elementary free group is a union of a chain of regular NTQ groups.

Recall that a non-standard model of arithmetic is any ring $\tilde{\mathbb{Z}}$ such that $\mathbb{Z} \equiv \tilde{\mathbb{Z}}$.

Theorem

For every non-standard countable model of arithmetic $\tilde{\mathbb{Z}}$ there exists a unique non-standard free group $F(\tilde{\mathbb{Z}})$ such that:

- F is the "standard part" of $F(\tilde{\mathbb{Z}})$;
- F is an elementary subgroup of $F(\tilde{\mathbb{Z}})$;
- not only $F \equiv F(\tilde{\mathbb{Z}})$, but F and $F(\tilde{\mathbb{Z}})$ are equivalent in the weak second order logic. They are strong models of the first-order theory of F.

Non-standard models of F

The following is interesting for us in this talk.

Fact

The groups $F(\tilde{\mathbb{Z}})$ are $\tilde{\mathbb{Z}}$-groups.

Moreover, the question below seems quite plausible.

Question

Is it true that $F(\tilde{\mathbb{Z}})$ is the free $\tilde{\mathbb{Z}}$-group $F^{\tilde{\mathbb{Z}}}$?